Non-Euclidean Support Vector Classifiers for Sparse Learning

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Outline

Machine Learning

- Given

$$D := \left\{ (\boldsymbol{x}_i, y_i) \right\}_{i=1}^{N} \subseteq \left(\boldsymbol{X} \times \{\pm 1\} \right)^N \subseteq \left(\mathbb{R}^n \times \{\pm 1\} \right)^N,$$

  our goal is to find a linear function $\boldsymbol{x}^\top \boldsymbol{\omega} + b$ such that for any i, $y_i(\boldsymbol{x}_i^\top \boldsymbol{\omega} + b) \geq 0$, where $\boldsymbol{\omega} \in \mathbb{R}^n$, $b \in \mathbb{R}$.
- Given a special *loss function* $L : \mathbb{R} \times \mathbb{R} \to [0, +\infty]$, it is done by solving

$$\min_{\boldsymbol{\omega} \in \mathbb{R}^n, b \in \mathbb{R}} \sum_{i=1}^{N} L(y_i, \boldsymbol{x}_i^\top \boldsymbol{\omega} + b).$$

- But it is an ill-posed problem.

Regularization and Sparsity

- Regularization is a crucial technique to prevent machine learning algorithms from over-fitting, it has the general form

$$\min_{\boldsymbol{\omega}\in\mathbb{R}^n, b\in\mathbb{R}} \sum_{i=1}^{N} L(y_i, \boldsymbol{x}_i^\top \boldsymbol{\omega} + b) + \lambda\|\boldsymbol{\omega}\|_2^2.$$

- The final target of regularization is to obtain a *sparse* result, meaning that as many of the components of the parameter have values of 0 as possible. It is always done by 1-norm regularization.
- Understanding regularization and sparsity can help us to dive deep into learning theorems. There are many aspects to explore them, such as *functional analysis, convex analysis, statistical learning*, etc..

Support Vector Machine Classifiers

- *Support Vector Machine Classifier (SVM classifier)* is by far one of the most successful binary-classification methods, it can finally be represented by

$$\min_{\boldsymbol{\omega}\in\mathbb{R}^n, b\in\mathbb{R}} \sum_{i=1}^{N}[1-y_i(\boldsymbol{x}_i^\top\boldsymbol{\omega}+b)]_+ + \lambda\|\boldsymbol{\omega}\|_2^2.$$

- The Euclidean distance used by the classical SVM classifier leads to 2-norm regularization.

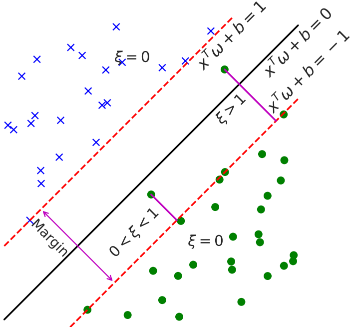


Figure *Support Vector Machine Classifiers*

Outline

Distances between Points and Hyperplanes

- Based on Theorem 2.2 in (O. L. Mangasarian, 1999), the distance derived from a general norm $\| \cdot \|$ from any points to a hyperplane

$$P := \{\boldsymbol{x} : \boldsymbol{x}^\top \boldsymbol{\omega} + b = 0 \, , \boldsymbol{x} \in \mathbb{R}^n\}$$

is given by

$$\text{dist}(\boldsymbol{x}, P) = \frac{|\boldsymbol{x}^\top \omega + b|}{\|\boldsymbol{\omega}\|_*},$$

where $\| \cdot \|_*$, defined as $\|\boldsymbol{z}\|_* = \sup\{\boldsymbol{z}^\top \boldsymbol{x} : \|\boldsymbol{x}\| < 1\}$, is the dual norm of $\| \cdot \|$.

Distances between Points and Hyperplanes: Examples

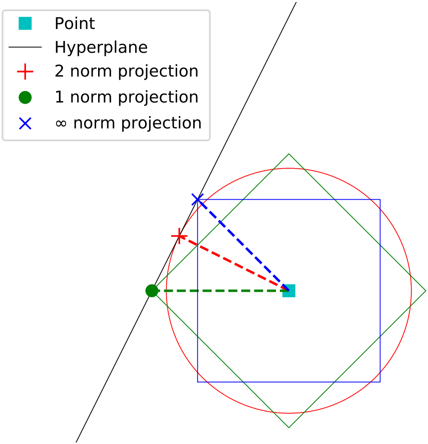


Figure Distances derived from special norms

- For 2 norm $\| \cdot \|_2$,

$$\mathrm{dist}(\boldsymbol{x}, P) = \frac{\|\boldsymbol{x}^\top \boldsymbol{\omega} + b\|}{\|\boldsymbol{\omega}\|_2}.$$

- For ∞ norm $\| \cdot \|_\infty$,

$$\mathrm{dist}(\boldsymbol{x}, P) = \frac{\|\boldsymbol{x}^\top \boldsymbol{\omega} + b\|}{\|\boldsymbol{\omega}\|_1}.$$

Non-Euclidean Support Vector Machine Classifiers

- The non-Euclidean SVM classifier has the form

$$
\begin{array}{lll}
\min\limits_{\boldsymbol{\omega}, b, \boldsymbol{\xi}} & \|\boldsymbol{\omega}\|_* + C\sum\limits_{i=1}^{N} \xi_i & \\
\text{subject to} & y_i(\boldsymbol{x}_i^T \boldsymbol{\omega} + b) \geq 1 - \xi_i, & \forall i, \\
& \xi_i \geq 0, & \forall i.
\end{array} \tag{1}
$$

  where C is the Lagrangian multiplier, $\boldsymbol{\xi} = \{\xi_i\}_{i=1}^{N}$ are slack variables.
- It is equivalent to the following unconstrained optimization problem

$$
\min_{\boldsymbol{\omega}, b} \underbrace{\sum_{i=1}^{N} [1 - y_i(\boldsymbol{x}_i^T \boldsymbol{\omega} + b)]_+}_{\text{Hinge Loss}} + \underbrace{\lambda \|\boldsymbol{\omega}\|_*}_{\text{Arbitrary Norm Regularization}} . \tag{2}
$$

Solutions for Optimization Problems

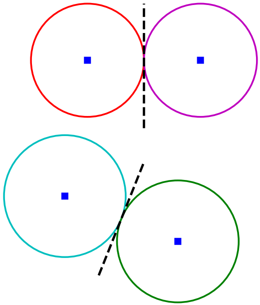
Proposition (Existences and Conditional Uniqueness of Solutions for Optimization Problems)

1. *Solution sets for constrained optimization problems (1), unconstrained optimization problem (2) are non-empty, compact and convex.*
2. *If $\|\cdot\|_*$ is strictly convex, constrained optimization problems (1), unconstrained optimization problem (2) have one and only one solution.*

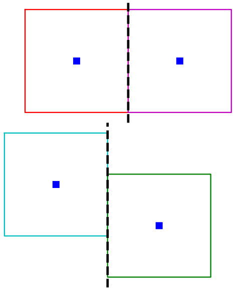
Theorem (Equivalence Between Solution Sets for Constrained and Unconstrained Optimization Problems)

1. *Suppose $\boldsymbol{v}^* = (\boldsymbol{\omega}^*, b^*, \boldsymbol{\xi}^*)$ is one of the solutions of (1), then $\boldsymbol{v}^* = (\boldsymbol{\omega}^*, b^*)$ is also one of the solutions of (2).*
2. *Suppose $\boldsymbol{v}^* = (\boldsymbol{\omega}^*, b^*)$ is one of the solutions of (2), then $\boldsymbol{v}^* = (\boldsymbol{\omega}^*, b^*, \boldsymbol{\xi}^*)$ is also one of the solutions of (1).*

Sparsity of 1-norm SVM Classifiers I



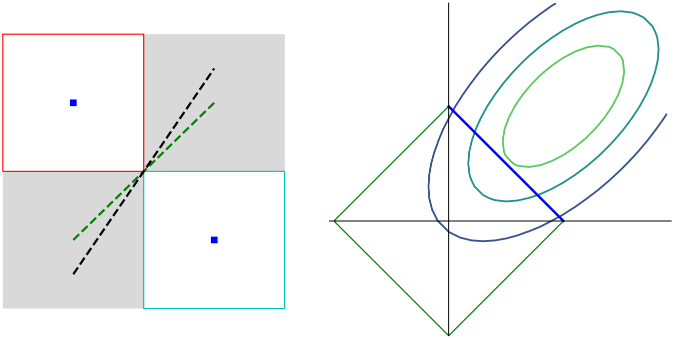
(a) 2 norm.



(b) ∞ norm.

Figure Maximal margin classifiers by 2 norm and ∞ norm in the case of 2 distinct points.

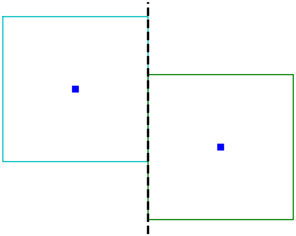
Sparsity of 1-norm SVM Classifiers II



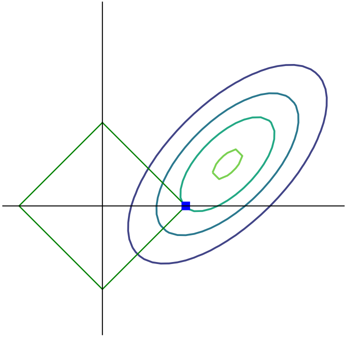
(a) Infinite solutions in data space. (b) Infinite solutions in parameter space.

Figure The geometric explanation for infinite solutions for the SVM classifier by ∞-norm margin.

Sparsity of 1-norm SVM Classifiers III



(a) Unique solution in data space.



(b) Unique solution in parameter space.

Figure The geometric explanation for unique solution for the SVM classifier by ∞-norm margin.

Outline

Summary

1. We extended the classical SVM classifiers to help us understand 1-norm regularization and provide a new view to study sparse learning.
2. We supplemented the mathematical backgrounds of 1-norm SVM classifiers.
3. We presented several special examples to show the sparsity by the 1-norm SVM classifiers.

Thank You!

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