Error Bounds for the Generalized Power Cone and Applications in Algebraic Structure

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Conic feasibility problems and error bounds

Conic feasibility problem: Let $\mathcal{K} \subset \mathbb{R}^n$ be a closed convex cone, $\mathcal{L}$ be a subspace of $\mathbb{R}^n$ and $\boldsymbol{a} \in \mathbb{R}^n$.

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$$\operatorname{dist}(\boldsymbol{x}, \mathcal{K} \cap (\mathcal{L} + \boldsymbol{a})) \le c_B(\max\{\operatorname{dist}(\boldsymbol{x}, \mathcal{K}), \operatorname{dist}(\boldsymbol{x}, \mathcal{L} + \boldsymbol{a})\})^{\theta}$$

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One-step facial residual functions-based approach: The framework based on facial reduction algorithm (Borwein, Wolkowicz ‘81) and one-step facial residual functions ($\mathbb{1}$-FRFs) (Lindstrom, Lourenço, Pong ‘22a) is theoretically adaptable for any closed convex cones (Lindstrom, Lourenço, Pong ‘22a ‘22b).

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Partial Polyhedral Slater's (PPS) condition: The PPS condition holds for $\{\mathcal{K}, \mathcal{L} + \boldsymbol{a}\}$ if one of the following conditions holds (Lourenço '21):

1. $\mathcal{K}$ is polyhedral;
2. $(\mathcal{L} + \boldsymbol{a}) \cap \operatorname{ri} \mathcal{K} \neq \emptyset$;
3. $\mathcal{K}$ can be written as $\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2$ where $\mathcal{K}^1$ is polyhedral and $(\mathcal{L} + \boldsymbol{a}) \cap (\mathcal{K}^1 \times \operatorname{ri} \mathcal{K}^2) \neq \emptyset$.

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Idea: Use the facial reduction algorithm to find a chain of faces:

$$\mathcal{F}_{\min} = \mathcal{F}_{\ell} \subsetneq \mathcal{F}_{\ell-1} \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

where $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{\boldsymbol{z}_i\}^{\perp}$ and $\boldsymbol{z}_i \in \mathcal{F}_i^* \cap \mathcal{L}^{\perp} \cap \{\boldsymbol{a}\}^{\perp}$ for $i = 1, \ldots, \ell - 1$, such that $\mathcal{F}_{\min}$ satisfies the PPS condition.
In each facial reduction step, use one-step facial residual functions to connect the "current" and "next" faces and compose them together to get the whole error bounds.

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Notation: The distance to the PPS condition $d_{\mathrm{PPS}}(\mathcal{K}, \mathcal{L} + \boldsymbol{a}) = \ell - 1$.

The generalized power cone

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$$\mathcal{P}_{m,n}^{\boldsymbol{\alpha}} = \left\{ \boldsymbol{x} = (\overline{\mathbf{x}}, \widetilde{\boldsymbol{x}}) \in \mathbb{R}^{m+n} \,\middle|\, \|\overline{\mathbf{x}}\| \leq \prod_{i=1}^{n} \widetilde{x}_i^{\alpha_i},\ \overline{\mathbf{x}} \in \mathbb{R}^m,\ \widetilde{\mathbf{x}} \in \mathbb{R}_+^n \right\},$$

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Observations:

- ✔ If $n = 1$, it is exactly a second-order cone.
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- ★ It is self-dual: A cone $\mathcal{K}$ is called self-dual if there exists a positive definite matrix Q such that $Q\mathcal{K} = \mathcal{K}^*$.

Motivations

1. The generalized power cones admit modeling of certain problems (Nesterov ‘12, Skajaa, Ye ‘15) and have found applications in geometric programs, generalized location problems, portfolio optimization, and nonnegativity problems (Chares ‘09, MOSEK ‘22, Skajaa, Ye ‘15, Murray ‘21).

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2. The generalized power cone is the final piece of the conic wheel (MOSEK ‘22), which contains five cones: $\mathbb{R}^n_+$, $\mathcal{Q}^{n+1}$, $\mathcal{S}^n_+$, $\mathcal{K}_{\exp}$, and $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$, and it is believed that (Lubin, Yamangil, Bent, Vielma ‘16, MINLPLib ‘22)

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3. The algebraic structure (automorphism group, homogeneity, reducibility, perfectness) of the generalized power cone is still an open problem except for the two symmetric cases and some particular cases, e.g., $m = 1$, $n = 2$ and $\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{2}{3}$ in (Truong, Tunçel ‘04).

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Facial structure and $\mathbb{1}$-FRFs of $\mathcal{P}_{m,n}^{\boldsymbol{\alpha}}$

Facial structure: Let $\boldsymbol{z} := (\overline{\mathbf{z}}, \widetilde{\mathbf{z}}) \in \partial(\mathcal{P}_{m,n}^{\boldsymbol{\alpha}})^* \setminus \{\mathbf{0}\}$.

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where $\beta := \sum_{i:\widetilde{z}_i > 0} \alpha_i$.

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Error bounds for $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$

Theorem 1. Consider the generalized power cone $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ and its dual cone $(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^*$. Let $\mathcal{L} \subseteq \mathbb{R}^{m+n}$ be a subspace and $\boldsymbol{a} \in \mathbb{R}^{m+n}$ be given. Suppose that $(\mathcal{L} + \boldsymbol{a}) \cap \mathcal{P}^{\boldsymbol{\alpha}}_{m,n} \neq \emptyset$. Then the following items hold.

1. $d_{\mathrm{PPS}}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}, \mathcal{L} + \boldsymbol{a}) \leq 1$.
2. If $d_{\mathrm{PPS}}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}, \mathcal{L} + \boldsymbol{a}) = 0$, then a Lipschitz error bound holds.
3. If $d_{\mathrm{PPS}}(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}, \mathcal{L} + \boldsymbol{a}) = 1$, consider the chain of faces $\mathcal{F} \subsetneq \mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ with length being 2.
   i. If $\mathcal{F} = \mathcal{F}_{\mathrm{r}}$, then a Hölderian error bound with exponent $\frac{1}{2}$ holds.
   ii. If $\mathcal{F} = \mathcal{F}_{\boldsymbol{z}}$ with $\boldsymbol{z} \in (\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})^* \cap \mathcal{L}^{\perp} \cap \{\boldsymbol{a}\}^{\perp}$, then a Hölderian error bound with exponent $\beta := \sum_{i:\tilde{z}_i > 0} \alpha_i$ holds.
   iii. If $\mathcal{F} = \{\mathbf{0}\}$, then a Lipschitz error bound holds.
4. All these error bounds are the best in the sense stated in (Lindstrom, Lourenço, Pong '22b).

Automorphisms of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$

Remark: When $n = 2$ and $\boldsymbol{\alpha} = (1/2, 1/2)$, $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ is isomorphic to the second-order cone, whose automorphism group is well-known.

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Theorem 2. For $m \geq 1, n > 2$ and any $\boldsymbol{\alpha} \in (0,1)^n$ such that $\sum_{i=1}^n \alpha_i = 1$, or for $m \geq 1, n = 2$ and any $\boldsymbol{\alpha} \in (0,1)^2$ such that $\alpha_1 \neq \alpha_2$ and $\alpha_1 + \alpha_2 = 1$, it holds that $A \in \mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$ if and only if

$$A = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix} \qquad \text{with } B \in \mathbb{R}^{m\times m}, E \in \mathbb{R}^{n\times n}$$

for some (invertible) generalized permutation matrix E with positive nonzero entries and invertible matrix B satisfying

$$\|B\boldsymbol{x}\| = \prod_{k=1}^{n} (E_{k,l_k})^{\alpha_{l_k}} \|\boldsymbol{x}\| \text{ for all } \boldsymbol{x} \in \mathbb{R}^m,$$

where E_{k,l_k} is the nonzero element in the k-th row of E and $\alpha_{l_k} = \alpha_k$.

Dimension of Aut $(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$

Theorem 3. Let $m \geq 1$, $n \geq 2$ and $\boldsymbol{\alpha} \in (0,1)^n$ such that $\sum_{i=1}^n \alpha_i = 1$, then we have the following statements about $\dim \mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$.

1. If $m \geq 1$, $n = 2$ and $\boldsymbol{\alpha} := (0.5, 0.5)$, then $\dim \mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = \frac{m^2+3m+4}{2}$.
2. If $m \geq 1$, $n > 2$ and $\sum_{i=1}^n \alpha_i = 1$ or $m \geq 1$, $n = 2$, $\alpha_1 \neq \alpha_2$ and $\alpha_1 + \alpha_2 = 1$, then the Lie algebra of $\mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$, denoted by $\mathrm{Lie\,Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$, is of form:

$$\mathrm{Lie\,Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = \left\{ \begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & \mathrm{diag}(\boldsymbol{h}) \end{bmatrix} \;\middle|\; \begin{array}{l} G + G^\top = 2\boldsymbol{\alpha}^\top \boldsymbol{h} I_m, \\ G \in \mathbb{R}^{m\times m},\ \boldsymbol{h} \in \mathbb{R}^n \end{array} \right\}.$$

Hence, $\dim \mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = \dim \mathrm{Lie\,Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = n + \frac{m(m-1)}{2}$.

Dimension of $\mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$

Theorem 3. Let $m \geq 1$, $n \geq 2$ and $\boldsymbol{\alpha} \in (0,1)^n$ such that $\sum_{i=1}^{n} \alpha_i = 1$, then we have the following statements about $\dim \mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$.

1. If $m \geq 1$, $n = 2$ and $\boldsymbol{\alpha} := (0.5, 0.5)$, then $\dim \mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = \frac{m^2+3m+4}{2}$.
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   Hence, $\dim \mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = \dim \mathrm{Lie}\,\mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}) = n + \frac{m(m-1)}{2}$.

Key ideas:

1. $\mathrm{Aut}\,(\mathcal{P}^{\boldsymbol{\alpha}}_{m,n})$, the automorphism group of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$, is a Lie group(a group that is also a differentiable manifold).
2. The Lie algebra associated with a Lie group is the tangent space of this Lie group at the identity element.
3. A Lie group and its Lie algebra share the same dimension.

Reducibility and perfectness

Reducibility: A cone is said to be reducible if it can be expressed as a direct sum of two nonempty and nontrivial cones.

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Perfectness: For a proper cone $\mathcal{K} \subseteq \mathbb{R}^p$, its complementarity set is defined as

$$C(\mathcal{K}) := \{(\boldsymbol{x}, \boldsymbol{s}) \mid \boldsymbol{x} \in \mathcal{K}, \boldsymbol{s} \in \mathcal{K}^*, \langle \boldsymbol{x}, \boldsymbol{s} \rangle = 0\}.$$

We say that $\mathcal{K}$ is perfect (Gowda, Tao '14) if there exist p linearly independent matrices $L_i \in \mathrm{Lie\,Aut}\,(\mathcal{K})$ such that

$$C(\mathcal{K}) = \{(\boldsymbol{x}, \boldsymbol{s}) \mid \boldsymbol{x} \in \mathcal{K}, \boldsymbol{s} \in \mathcal{K}^*, \langle L_i \boldsymbol{x}, \boldsymbol{s} \rangle = 0 \quad \forall i\}.$$

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✍️ If a cone is perfect, the dual problem of the corresponding conic linear programming can be written as a complementarity problem with a square system, then some specific algorithms can be applied.

The algebraic structure of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$

Corollary 1. Let $m \geq 1$, $n \geq 2$ and $\boldsymbol{\alpha} \in (0,1)^n$ such that $\sum_{i=1}^{n} \alpha_i = 1$, then the following statements hold for $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$.

1. $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ is irreducible.
2. If $m \geq 1, n = 2$ and $\boldsymbol{\alpha} := (0.5, 0.5)$, then $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ is homogeneous and perfect.
3. If $m \geq 1$, $n > 2$ and $\sum_{i=1}^{n} \alpha_i = 1$ or $m \geq 1$, $n = 2$, $\alpha_1 \neq \alpha_2$ and $\alpha_1 + \alpha_2 = 1$, then $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ is nonhomogeneous. If $1 \leq m \leq 2$, then $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ is not perfect. If $m \geq 3$, then $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ is perfect.

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Key ideas:

1. If a closed convex pointed cone $\mathcal{K}$ is reducible, i.e., $\mathcal{K}$ is a direct sum of two nonempty, nontrivial sets $\mathcal{K}_1$ and $\mathcal{K}_2$, then we have $\mathcal{K}_1 \ntrianglelefteq \mathcal{K}$, $\mathcal{K}_2 \ntrianglelefteq \mathcal{K}$ and $\dim(\mathcal{K}) = \dim(\mathcal{K}_1) + \dim(\mathcal{K}_2)$.
2. A cone $\mathcal{K}$ is homogeneous if for every $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{ri} \mathcal{K}$, there is a matrix A such that $A\boldsymbol{x} = \boldsymbol{y}$ and $A\mathcal{K} = \mathcal{K}$.
3. A proper cone $\mathcal{K}$ is perfect if and only if $\dim\mathrm{LieAut}(\mathcal{K}) \geq \dim(\mathcal{K})$.

Conclusion

- The error bounds for $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ are completely established.
- The first result regarding the automorphism group of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$.
- The first rigorous proof of the nonhomogeneity of $\mathcal{P}^{\boldsymbol{\alpha}}_{m,n}$ in the general case.
- An interesting example of a set of cones that is proved to be self-dual, irreducible, nonhomogeneous and perfect simultaneously.

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🤠 Thanks for listening! 👋